

Gain Uncertainty of Arrayed Antennas

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The gain uncertainty of an array of antennas is derived in terms of the gain uncertainty of the individual elements. For the case where the gain uncertainties of the individual elements are about equal, the gain uncertainty (uncorrelated errors) of the array is less than that of the individual elements. In the DSN, the gain uncertainty of an array composed of a 64-meter antenna and one or more 34-meter antennas is most sensitive to the uncertainty of the gain of the 64-meter antenna. For example, a 64-meter antenna ($G \cong 61.7 \pm 0.1$ dB) and a 34-meter antenna ($G_2 \cong 56.1 \pm 0.3$ dB) result in an array with $G \cong 62.8 \pm 0.1$ dB (uncorrelated), ± 0.15 dB (correlated). For this example, the array gain uncertainty is not significantly affected by the smaller element gain tolerance.

I. Introduction

When a 64-meter and one or more 34-meter DSN antennas are arrayed, the increase in the gain of the array relative to the gain of the 64-meter antenna is of the same order of magnitude (in dB) as the uncertainty in the gain of each of the antennas. With this in mind, it might appear that only minimal, if any, operational advantage can be gained by arraying, since conservative telecommunications link design requires one to consider the sum of the negative tolerances. This report shows that arraying is valuable to the link performance even under the conditions stated. It is recognized that many factors go into making up the performance of an antenna array. This analysis only addresses the performance uncertainty of the array due to the uncertainty in the gain of the individual antennas.

$$G = \sum_{i=1}^n G_i \quad (1)$$

and (assuming statistical independence) the uncorrelated array gain uncertainty is

$$\sigma_u = \left[\sum_{i=1}^n (\sigma_i)^2 \left(\frac{\partial G}{\partial G_i} \right)^2 \right]^{1/2} \quad (2)$$

but

$$\frac{\partial G}{\partial G_1} = \frac{\partial G}{\partial G_2} = \dots = \frac{\partial G}{\partial G_n} = 1$$

Therefore

$$\sigma_u = \left[\sum_{i=1}^n (\sigma_i)^2 \right]^{1/2} \quad (3)$$

II. Theory

Consider a set of antennas with gains G_1, G_2, \dots, G_n and uncertainties $\sigma_1, \sigma_2, \dots, \sigma_n$ respectively. The array gain is given by

and (assuming statistical dependence) the correlated array gain uncertainty is or

$$\sigma_c = \sum_{i=1}^n \sigma_i \quad (4)$$

where σ_u assumes that the σ_i s are uncorrelated and σ_c assumes that the σ_i s are fully correlated and add in the worst possible manner. Note that the above relationships hold when the gains and uncertainties are expressed as ratios. Since

$$G = 10^{G_{dB}/10} \quad (5)$$

and

$$\sigma = \sigma_{dB} \left(\frac{\partial G}{\partial G_{dB}} \right) \quad (6)$$

we get

$$\sigma = \frac{\ln 10}{10} (G) (\sigma_{dB}) \quad (7)$$

or

$$\sigma_{dB} = \frac{10}{\ln 10} \left(\frac{\sigma}{G} \right) \quad (8)$$

substituting (1) and (3) into (8) gives

$$\sigma_{u dB} = \frac{10}{\ln 10} \frac{\left[\sum_{i=1}^n (\sigma_i)^2 \right]^{1/2}}{\sum_{i=1}^n G_i} \quad (9)$$

or

$$\sigma_{u dB} = \frac{\left[\sum_{i=1}^n (G_i \sigma_{i dB})^2 \right]^{1/2}}{\sum_{i=1}^n G_i} \quad (10)$$

while substituting (1) and (4) into (8) gives

$$\sigma_{c dB} = \frac{10}{\ln 10} \frac{\sum_{i=1}^n \sigma_i}{\sum_{i=1}^n G_i} \quad (11)$$

$$\sigma_{c dB} = \frac{\sum_{i=1}^n G_i \sigma_{i dB}}{\sum_{i=1}^n G_i} \quad (12)$$

III. Example

Let us array two antennas with gains of 61.7 and 56.1 dB and assume that both antennas are known to 0.1 dB one sigma. This approximates the DSN 64-meter and 34-meter antennas operating at S-band. Letting the subscript 1 denote the 64-meter antenna and 2 the 34-meter antenna, we get from Eq. (1)

$$G = G_1 + G_2 = 10^{(61.70/10)} + 10^{(56.10/10)} = 10^{(6.276)}$$

and from Eqs. (3) and (7)

$$\begin{aligned} \sigma_u &= (\sigma_1^2 + \sigma_2^2)^{1/2} \\ &= \left\{ \left[\frac{\ln 10}{100} (10^{6.17}) \right]^2 + \left[\frac{\ln 10}{100} (10^{5.61}) \right]^2 \right\}^{1/2} \\ &= 35,326 \end{aligned}$$

Therefore, the gain of this two-antenna array is

$$G_{dB} = 62.76 \pm 0.081 \text{ dB (one sigma}_u\text{)}$$

or if we consider σ_c ,

$$\begin{aligned} \sigma_c &= \sigma_1 + \sigma_2 \\ &= \frac{\ln 10}{100} (10^{6.17}) + \frac{\ln 10}{100} (10^{5.61}) \\ &= 43,438 \end{aligned}$$

we get

$$G_{dB} = 62.76 \pm 0.100 \text{ dB (one sigma}_c\text{)}$$

We note that in this example, the gain of the array increased by 1.06 dB relative to the gain of the 64-meter antenna. However, even in the fully correlated error worst case, the uncertainty in the array gain has not increased over that of the individual antennas!

Figure 1 is a plot of the statistical (σ_u) and worst case (σ_c) gain error of this array with σ_1 held constant at 0.1 dB and σ_2 allowed to vary between 0 and 1 dB. From Fig. 1 we see that σ_2 can go to approximately 0.3 dB before the formal error of this two-antenna array reaches $\sigma_u = 0.1$ dB. From Fig. 1 we also see that even in the "worst case" the slope of the uncertainty is less than 1. That is,

$$\frac{d\sigma_c}{d\sigma_2} < 1$$

for

$$\sigma_1 = 0.1 \text{ dB}$$

This demonstrates that it is more critical to know the gain of the larger antenna of this two-element array. Indeed, it would be expected that the largest (highest gain) antenna in an array of unequal gain antennas would contribute most to the total array gain as well as contributing most to the uncertainty of the total array gain.

In Fig. 2, we see plotted the sum of the gains of a 64-meter (61.7-dB) antenna and zero to three 34-meter (56.1-dB) antennas. All antennas are assumed known to 0.1-dB one sigma. In this figure we see the uncorrelated error bars shrink as the number of antennas in the array is increased while the correlated error bars remain constant and the mean gain increases.

IV. Another Example

Let us look at the case of arraying a 64-m antenna (61.7 \pm 0.1 dB gain at S-band) with a second antenna whose gain will be allowed to vary downward from 61.7 dB, the uncertainty in the gain of the second antenna being kept constant at 0.1 dB. That is,

$$G_1 = 61.7 \pm 0.1 \text{ dB}$$

and

$$G_2 \leq 61.7 \pm 0.1 \text{ dB}$$

We now calculate that fraction of the uncertainty in the array gain due to the uncertainty in the gain of the smaller antenna. Looking at the uncorrelated case first, we get:

$$U = \frac{\sigma_2}{\sigma_u} = \frac{\sigma_2}{(\sigma_1^2 + \sigma_2^2)^{1/2}}$$

$$= \frac{(10^{G_{2dB}/10}) (\sigma_{2dB})}{\left\{ \left[(10^{G_{1dB}/10}) (\sigma_{1dB}) \right]^2 + \left[(10^{G_{2dB}/10}) (\sigma_{2dB}) \right]^2 \right\}^{1/2}}$$

Setting

$$G_{1dB} = 61.7$$

and

$$\sigma_{1dB} = \sigma_{2dB} = 0.1,$$

we get that

$$U = \frac{10^{G_{2dB}/10}}{\left[(10^{61.7})^2 + (10^{G_{2dB}/10})^2 \right]^{1/2}} \quad (13)$$

In the correlated case, we can write that

$$C = \frac{\sigma_2}{\sigma_c} = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$

$$= \frac{(10^{G_{2dB}/10}) (\sigma_{2dB})}{(10^{G_{1dB}/10}) (\sigma_{1dB}) + (10^{G_{2dB}/10}) (\sigma_{2dB})}$$

Again,

$$G_{1dB} = 61.7$$

and

$$\sigma_{1dB} = \sigma_{2dB} = 0.1$$

Then

$$C = \frac{10^{G_{2dB}/10}}{10^{6.17} + 10^{G_{2dB}/10}} \quad (14)$$

Equations (13) and (14) are shown plotted in Fig. 3 as a function of $G_1 - G_2$. In Fig. 3 we notice the apparently anomalous situation of U reaching 0.707 (instead of 0.5) when $G_1 = G_2$. This result is caused in the uncorrelated case because the

variances (σ_i^2) add while in the correlated case the standard deviations (σ_i) add.

V. Conclusion

From the preceding we see that (1) the array gain increases as antennas are added to the array, (2) the error bars do not grow under the constraint that we know the performance of all elements equally well, and (3) the knowledge of the 64-meter antenna performance is more important to the array than knowledge of the 34-meter antennas and therefore, we should spend most of our effort quantifying the larger antenna.

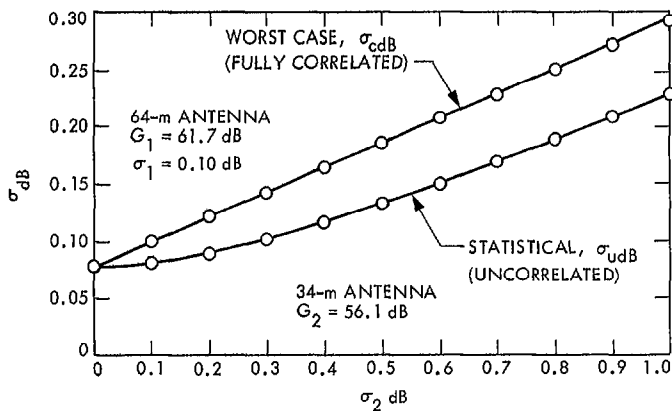


Fig. 1. Uncertainty in the array gain of a 64-meter antenna and a 34-meter antenna

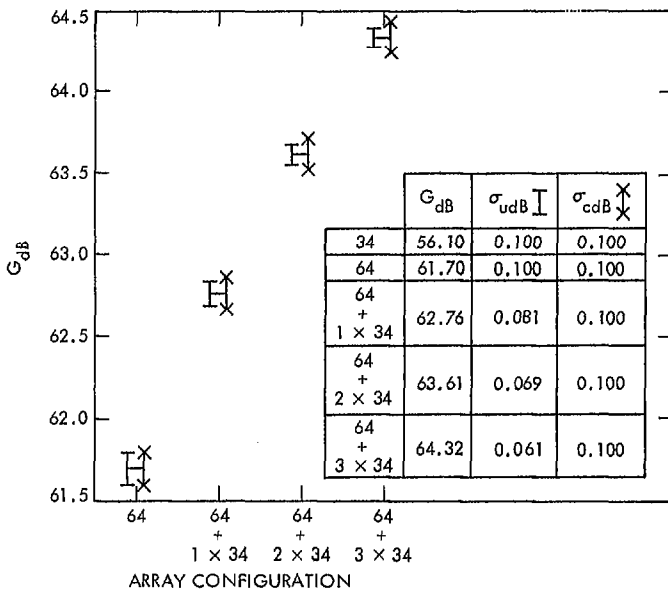


Fig. 2. Array gain vs configuration

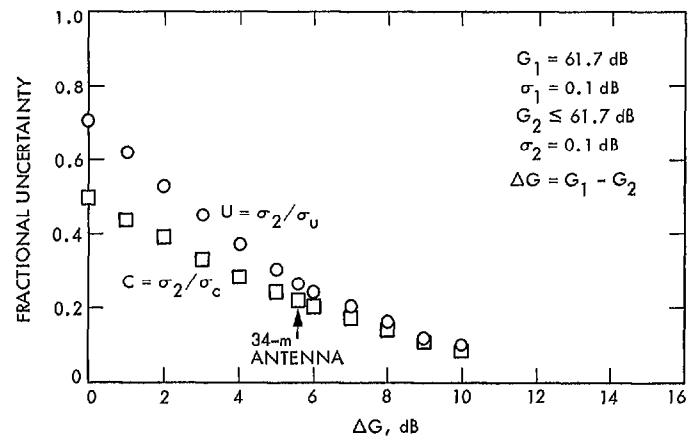


Fig. 3. Fractional uncertainty in array gain due to smaller antenna